

Conformally invariant differential operators on tensor densities

V. Ovsienko [‡] P. Redou [§]

Abstract

Let \mathcal{F}_λ be the space of tensor densities on \mathbb{R}^n of degree λ (or, equivalently, of conformal densities of degree $-\lambda n$) considered as a module over the Lie algebra $\mathfrak{o}(p+1, q+1)$. We classify $\mathfrak{o}(p+1, q+1)$ -invariant bilinear differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to \mathcal{F}_ν . The classification of linear $\mathfrak{o}(p+1, q+1)$ -invariant differential operators from \mathcal{F}_λ to \mathcal{F}_μ already known in the literature (see [6, 9]) is obtained in a different manner.

Keywords: Conformal structures, modules of differential operators, tensor densities, invariant differential operators.

[‡]CNRS, Centre de Physique Théorique, CPT-CNRS, Luminy Case 907, F-13288 Marseille, Cedex 9, FRANCE; <mailto:ovsienko@cpt.univ-mrs.fr>

[§]Ecole Nationale d'Ingenieurs de Brest Technopole Brest Iroise - Parvis Blaise Pascal BP 30815 - 29608 Brest Cedex - FRANCE; <mailto:redou@enib.fr>

1 Introduction

1.1 Linear and bilinear SL_2 -invariant operators

In the one-dimensional case, the problem of classification of SL_2 -invariant (bi)linear differential operators has already been treated in the classical literature.

Consider the action of $\mathrm{SL}(2, \mathbb{R})$ on the space of functions in one variable, say on $C^\infty(\mathbb{RP}^1)(\cong C^\infty(S^1))$ given by

$$f(x) \mapsto f\left(\frac{ax+b}{cx+d}\right)(cx+d)^{-2\lambda} \quad (1.1)$$

depending on a parameter $\lambda \in \mathbb{R}$ (or \mathbb{C}). This $\mathrm{SL}(2, \mathbb{R})$ -module is called the space of tensor densities of degree λ and denoted \mathcal{F}_λ . The classification of $\mathrm{SL}(2, \mathbb{R})$ -invariant linear differential operators from \mathcal{F}_λ to \mathcal{F}_μ (i.e. of the operators commuting with the action (1.1)) was obtained in classical works on projective differential geometry. The result is as follows: there exists a unique (up to a constant) $\mathrm{SL}(2, \mathbb{R})$ -invariant linear differential operator

$$A_k : \mathcal{F}_{\frac{1-k}{2}} \rightarrow \mathcal{F}_{\frac{1+k}{2}} \quad (1.2)$$

where $k = 0, 1, 2, \dots$, and there are no $\mathrm{SL}(2, \mathbb{R})$ -invariant operators from \mathcal{F}_λ to \mathcal{F}_μ for any other values of λ and μ . Each operator A_k is of order k and, for any choice of the coordinate x such that the $\mathrm{SL}(2, \mathbb{R})$ -action is as in (1.1), is plainly given by the k -th order derivative, $(A_k f)(x) = f^{(k)}(x)$.

The bilinear $\mathrm{SL}(2, \mathbb{R})$ -invariant differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to \mathcal{F}_ν were already classified by Gordan [8]. For generic values of λ and μ (more precisely, for $\lambda, \mu \neq 0, \frac{1}{2}, 1, \dots$), there exists a unique (up to a constant) $\mathrm{SL}(2, \mathbb{R})$ -invariant bilinear differential operator

$$B_k : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+k} \quad (1.3)$$

where $k = 0, 1, 2, \dots$, and there are no $\mathrm{SL}(2, \mathbb{R})$ -invariant operators for any other value of ν . This differential operator is given by the formula

$$B_k(f, g) = \sum_{i+j=k} (-1)^i \binom{2\mu+k-1}{i} \binom{2\lambda+k-1}{j} f^{(i)} g^{(j)} \quad (1.4)$$

and is called the transvectant.

1.2 Multi-dimensional analogues

In the multi-dimensional case, one has to distinguish the conformally flat case, that can be reduced to \mathbb{R}^n endowed with the standard $\mathfrak{o}(p+1, q+1)$ -action, where $n = p + q$, and the curved (generic) case of an arbitrary manifold M endowed with a conformal structure.

In the conformally flat case, the analogues of the operators (1.2) was classified in [6]. The result is as follows.

Theorem 1.1. ([6]) *There exists a unique (up to a constant) $\mathfrak{o}(p+1, q+1)$ -invariant linear differential operator*

$$A_{2k} : \mathcal{F}_{\frac{n-2k}{2n}} \rightarrow \mathcal{F}_{\frac{n+2k}{2n}} \quad (1.5)$$

where $k = 0, 1, 2, \dots$, and there are no $\mathfrak{o}(p+1, q+1)$ -invariant operators from \mathcal{F}_λ to \mathcal{F}_μ for any other values of λ and $\mu \in \mathbb{R}$ (or \mathbb{C}).

In the adopted coordinate system (corresponding to the chosen conformally flat structure), the explicit expressions of the operators A_{2k} are

$$A_{2k} = \Delta^k, \quad \text{where} \quad \Delta = g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \quad (1.6)$$

In the generic (curved) case, the situation is much more complicated, see [9, 10]. (We also refer to [2] for a recent study of conformally invariant differential operators on tensor fields, and a complete list of references.)

The purpose of this note is to extend the classical Gordan result to the multi-dimensional case. We will classify $\mathfrak{o}(p+1, q+1)$ -invariant bilinear differential operators consider differential operators on tensor densities on \mathbb{R}^n , where $n = p + q$. The results can be also generalized for an arbitrary manifold M endowed with a conformally flat structure (e.g. upon a pseudo-Riemannian manifold (M, g) with a conformally flat metric). We will also give a simple direct proof of Theorem 1.6.

We will not consider the curved case and state here a problem of existence of bilinear conformally invariant differential operators for an arbitrary conformal structure.

Remark 1.2. There are other ways to generalize \mathfrak{sl}_2 -symmetries in the multi-dimensional case. For instance, one can consider the $\mathfrak{sl}(n+1, \mathbb{R})$ -action on \mathbb{R}^n ; this is related to projective differential geometry.

2 Main result

2.1 Modules of differential operators

Let \mathcal{F}_λ be the space of tensor densities of degree λ on \mathbb{R}^n , i.e. of smooth section of the line bundle $\Delta_\lambda(\mathbb{R}^n) = |\Lambda^n T^*\mathbb{R}^n|^{\otimes \lambda}$ over \mathbb{R}^n . We will be considering the space $\mathcal{D}_{\lambda,\mu}$ of linear differential operators from \mathcal{F}_λ to \mathcal{F}_μ and the space $\mathcal{D}_{\lambda,\mu;\nu}$ of bilinear differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to \mathcal{F}_ν . These spaces of differential operators are naturally $\text{Diff}(\mathbb{R}^n)$ - and $\text{Vect}(\mathbb{R}^n)$ -modules. Note that the modules $\mathcal{D}_{\lambda,\mu}$ have been studied in a series of recent papers (see [5] and references therein).

Denote g the standard quadratic form on \mathbb{R}^n of signature $p - q$, where $p + q = n$. The Lie algebra of infinitesimal conformal transformations is generated by the vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i} \\ X_{ij} &= x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i} \\ X_0 &= x^i \frac{\partial}{\partial x^i} \\ \bar{X}_i &= x_j x^j \frac{\partial}{\partial x^i} - 2x_i x^j \frac{\partial}{\partial x^j} \end{aligned} \tag{2.1}$$

where (x^1, \dots, x^n) are coordinates on \mathbb{R}^n and $x_i = g_{ij}x^j$, throughout this paper, sum over repeated indices is understood. Let us also consider the following Lie subalgebras

$$\mathfrak{o}(p, q) \subset \mathfrak{e}(p, q) \subset \mathfrak{o}(p + 1, q + 1) \subset \text{Vect}(\mathbb{R}^n) \tag{2.2}$$

where $\mathfrak{o}(p, q)$ is generated by the X_{ij} and the Euclidean subalgebra $\mathfrak{e}(p, q)$ by X_{ij} and X_i .

We will study the spaces $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{D}_{\lambda,\mu;\nu}$ as $\mathfrak{o}(p + 1, q + 1)$ -modules and classify the differential operators commuting with the $\mathfrak{o}(p + 1, q + 1)$ -action.

It should be stressed that the classification of differential operators invariant with respect to the Lie subalgebras $\mathfrak{o}(p, q)$ is the classical result of the Weyl theory of invariants [16] (see also [5] for the case of $\mathfrak{e}(p, q)$). We will use the Weyl classification in our work.

Remark 2.1. It is worth noticing that the conformal Lie algebra $\mathfrak{o}(p + 1, q + 1)$ is maximal in the class of finite-dimensional subalgebras of $\text{Vect}(\mathbb{R}^n)$, that is, any bigger subalgebra of $\text{Vect}(\mathbb{R}^n)$ is infinite-dimensional (see [1] for a simple proof).

2.2 Introducing multi-dimensional transvectants

The multi-dimensional analogues of the transvectants (1.4) are described in the following

Theorem 2.2. *For every $\lambda, \mu \neq 0, \frac{1}{n}, \frac{2}{n}, \dots$, there exists a unique (up to a constant) $\mathfrak{o}(p+1, q+1)$ -invariant bilinear differential operator*

$$B_{2k} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+\frac{2k}{n}} \quad (2.3)$$

where $k = 0, 1, 2, \dots$, and there are no $\mathfrak{o}(p+1, q+1)$ -invariant operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to \mathcal{F}_ν for any other value of ν .

The explicit formula for the operators B_{2k} is complicated and is known only in some particular cases.

Remark 2.3. The differential operators A_{2k} and B_{2k} are of order $2k$; comparing with the one-dimensional case, one has twice less invariant differential operators. Note that if one takes semi-integer k in (1.6), then the corresponding operator is pseudo-differential.

It would be interesting to obtain a complete classification of $\mathfrak{o}(p+1, q+1)$ -invariant bilinear differential operators (see [7] for the one-dimensional case).

3 Proof of Theorems 1.1 and 2.2

We will start the proof with classical results of the theory of invariants and describe the differential operators invariant with respect to the action of the Lie algebra $\mathfrak{e}(p, q)$. We refer [16] as a classical source and [5] for the description of the Euclidean invariants.

3.1 Euclidean invariants

Using the standard affine connection on \mathbb{R}^n , one identifies the space of linear differential operators on \mathbb{R}^n with the corresponding space of symbols, i.e., with the space of smooth functions on $T^*\mathbb{R}^n \cong \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ polynomial on $(\mathbb{R}^n)^*$. This identification is an isomorphism of modules over the algebra of affine transformations and allows us to apply the theory of invariants.

Moreover, choosing a (dense) subspace of symbols which are also polynomials on the first summand, one reduces the classification of $e(p, q)$ -invariant differential operators from $\mathcal{D}_{\lambda, \mu}$ to the classification of $e(p, q)$ -invariant polynomials in the space $\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n]$, where (ξ_1, \dots, ξ_n) are the coordinates on $(\mathbb{R}^n)^*$ dual to (x^1, \dots, x^n) .

Consider first invariants with respect to $o(p, q) \subset e(p, q)$. It is well-known (see [16]) that the algebra of $o(p, q)$ -invariant polynomials is generated by three elements

$$R_{xx} = g_{ij} x^i x^j, \quad R_{x\xi} = x^i \xi_i, \quad R_{\xi\xi} = g^{ij} \xi_i \xi_j \quad (3.1)$$

Second, taking into account the invariance with respect to translations in $e(p, q)$, any $e(p, q)$ -invariant polynomial $P(x, \xi)$ satisfies $\partial P / \partial x^i = 0$. The only remaining generator is $R_{\xi\xi}$ and, therefore, $e(p, q)$ -invariant linear differential operators from \mathcal{F}_λ to \mathcal{F}_μ are linear combinations of operators (1.6).

Note that the obtained result is, of course, independent on λ and μ since the degree of tensor densities does not intervene in the $e(p, q)$ -action.

3.2 Proof of Theorem 1.1

We must check now for which values of λ and μ the operators (1.6) from \mathcal{F}_λ to \mathcal{F}_μ are invariant with respect to the action of the full conformal algebra $o(p+1, q+1)$.

By definition, the action of a vector field X on an element $A \in \mathcal{D}_{\lambda, \mu}$ is given by

$$L_X^{\lambda, \mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda \quad (3.2)$$

where L_X^λ is the operator of Lie derivative of λ -density, namely

$$L_X^\lambda = X^i \frac{\partial}{\partial x^i} + \lambda \partial_i X^i \quad (3.3)$$

for any coordinate system.

Consider the action of the generator X_0 in (2.1) on the operator $A = \sum_{k \geq 0} c_k R_{\xi\xi}^k$. Using the preceding expressions, one readily gets

$$L_{X_0}^{\lambda, \mu}(A) = \sum_{i \geq 0} (n\delta - 2k) c_k R_{\xi\xi}^k \quad (3.4)$$

where $\delta = \mu - \lambda$. Thus, the invariance condition $L_{X_0}^{\lambda, \mu}(A) = 0$ is satisfied if and only if for each k in the above sum either $c_k = 0$ or $\delta = \frac{2k}{n}$; and one gets the values of the shift δ in accordance with (1.6).

Consider, at last, the action of the generators \bar{X}_i (with $i = 1, \dots, n$). After the identification of the differential operators with polynomials one has the following result from [5].

Proposition 3.1. *The action of the generator \bar{X}_i on $\mathcal{D}_{\lambda,\mu}$ is as follows*

$$L_{\bar{X}_i}^{\lambda,\mu} = L_{\bar{X}_i}^\delta - \xi_i T + 2(\mathcal{E} + n\lambda) \partial_{\xi_i} \quad (3.5)$$

where

$$L_{\bar{X}_i}^\delta = x_j x^j \partial_i - 2x_i x^j \partial_j - 2(\xi_i x_j - \xi_j x_i) \partial_{\xi_j} + 2\xi_j x^j \partial_{\xi_i} - 2n\delta x_i \quad (3.6)$$

is the cotangent lift, and where $T = \partial_{\xi_j} \partial_{\xi_j}$ is the trace and $\mathcal{E} = \xi_j \partial_{\xi_j}$ the Euler operator.

Applying $L_{\bar{X}_i}^{\lambda,\mu}$ to the operator $R_{\xi\xi}^k$ one then obtains

$$\begin{aligned} L_{\bar{X}_i}^{\lambda,\mu}(R_{\xi\xi}^k) &= 2(2k - n\delta)x_i R_{\xi\xi}^k \\ &\quad + 2k(n(2\lambda - 1) + 2k)\xi_i R_{\xi\xi}^{k-1} \end{aligned}$$

The first term in this expression vanishes for $2k - n\delta=0$, this condition is precisely the preceding one; the second term vanishes if and only if

$$\lambda = \frac{n - 2k}{2n}.$$

Hence the result.

3.3 Proof of Theorem 2.2

As in Section 3.1, let us first consider the operators invariant with respect to the Lie algebra $\mathfrak{e}(p, q)$. Again, identifying the bilinear differential operators with their symbols, one is led to study the algebra of $\mathfrak{e}(p, q)$ -invariant polynomials in the space $\mathbb{C}[x^1, \dots, x^n, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n]$. The Weyl invariant theory just applied guarantees that there are three generators:

$$R_{\xi\xi} = g^{ij} \xi_i \xi_j, \quad R_{\xi\eta} = g^{ij} \xi_i \eta_j, \quad R_{\eta\eta} = g^{ij} \eta_i \eta_j \quad (3.7)$$

Any $\mathfrak{e}(p, q)$ -invariant bilinear differential operator is then of the form

$$B = \sum_{r,s,t \geq 0} c_{rst} R^{r,s,t}$$

where, to simplify the notations, we put $R^{r,s,t} = R_{\xi\xi}^r R_{\xi\eta}^s R_{\eta\eta}^t$.

The action of a vector field X on a bilinear operator $B : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_\nu$ is defined as follows

$$(L_X^{\lambda,\mu;\nu} B)(f, g) = L_X^\nu B(f, g) - B(L_X^\lambda f, g) - B(f, L_X^\mu g) \quad (3.8)$$

Let us apply the generator X_0 to the operator B , one has

$$L_{X_0}^{\lambda,\mu;\nu}(B) = \sum_{r,s,t \geq 0} (n\delta - 2(r+s+t)) c_{rst} R^{r,s,t} \quad (3.9)$$

where $\delta = \nu - \mu - \lambda$. The equation $L_{X_0}^{\lambda,\mu}(B) = 0$ leads to the homogeneity condition

$$\delta = \frac{2(r+s+t)}{n} \quad (3.10)$$

The general expression for B retains the form

$$B_{2k} = \sum_{r+s+t=k} c_{rst} R^{r,s,t} \quad (3.11)$$

The operator B is of order $2k$ and $\nu = \lambda + \mu + 2k/n$.

Now, to determine the coefficients c_{rst} , one has to apply the generators \bar{X}_i . One has the following analog of Proposition 3.1.

Proposition 3.2. *The action of the generator \bar{X}_i on $\mathcal{D}_{\lambda,\mu;\nu}$ is given by*

$$L_{\bar{X}_i}^{\lambda,\mu;\nu} = L_{\bar{X}_i}^\delta - \xi_i T_\xi - \eta_i T_\eta + 2((\mathcal{E}_\xi + n\lambda) \partial_{\xi^i} + (\mathcal{E}_\eta + n\mu) \partial_{\eta^i}) \quad (3.12)$$

where

$$\begin{aligned} L_{\bar{X}_i}^\delta &= x_j x^j \partial_i - 2x_i x^j \partial_j - 2n\delta x_i \\ &\quad - 2((\xi_i x_j - \xi_j x_i) \partial_{\xi_j} + (\eta_i x_j - \eta_j x_i) \partial_{\eta_j}) \\ &\quad + 2(\xi_j x^j \partial_{\xi^i} + \eta_j x^j \partial_{\eta^i}) \end{aligned} \quad (3.13)$$

is just the natural lift of \bar{X}_i to $T^*\mathbb{R}^n \oplus T^*\mathbb{R}^n$.

Applying $L_{\bar{X}_i}^{\lambda,\mu;\nu}$ to each monomial, $R^{r,s,t} = R_{\xi\xi}^r R_{\xi\eta}^s R_{\eta\eta}^t$, in the operator B_{2k} , one immediately gets

$$\begin{aligned} L_{\bar{X}_i}^{\lambda,\mu;\nu}(R^{r,s,t}) &= 2(2k - n\delta) x_i R^{r,s,t} \\ &\quad + \left(2r(2r + n(2\lambda - 1)) R^{r-1,s,t} - s(s-1) R^{r,s-2,t+1} \right. \\ &\quad \left. + 2s(s+2t + n\mu - 1) R^{r,s-1,t} \right) \xi_i \\ &\quad + \left(2t(2t + n(2\mu - 1)) R^{r,s,t-1} - s(s-1) R^{r+1,s-2,t} \right. \\ &\quad \left. + 2s(s+2r + n\lambda - 1) R^{r,s-1,t} \right) \eta_i \end{aligned} \quad (3.14)$$

At last, applying $L_{\bar{X}_i}^{\lambda, \mu; \nu}$ to the operator B_{2k} written in the form (3.11) and collecting the terms, one readily gets the following recurrent system of two linear equations

$$\begin{aligned} & 2(r+1)(2(r+1) + n(2\lambda - 1)) c_{r+1, s, t} \\ & - (s+2)(s+1) c_{r, s+2, t-1} \\ & + 2(s+1)(s+2t + n\mu) c_{r, s+1, t} = 0 \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} & 2(t+1)(2(t+1) + n(2\mu - 1)) c_{r, s, t+1} \\ & - (s+2)(s+1) c_{r-1, s+2, t} \\ & + 2(s+1)(s+2r + n\lambda) c_{r, s+1, t} = 0 \end{aligned} \quad (3.16)$$

for the coefficients. For $\lambda, \mu \neq 0, \frac{1}{n}, \frac{2}{n}, \dots$ this system has a unique (up to a constant) solution. Indeed, choosing $c_{0, k, 0}$ as a parameter, one uses the first equation to express $c_{r+1, s, t}$ from $c_{r, *, *}$ and the second one to express $c_{r, s, t+1}$ from $c_{*, *, t}$.

Theorem 2.2 is proved.

The differential operators (1.2) and (1.3) play important rôle in the theory of modular functions (see [15, 3]), in projective differential geometry (see [17, 11]) and in the representation theory of $\mathrm{SL}(2, \mathbb{R})$. The transvectants have been recently used in [4, 13] to construct $\mathrm{SL}(2, \mathbb{R})$ -invariant star-products on T^*S^1 . We plan to discuss the relation of the conformally invariant operators described in this note to the representation theory in a subsequent paper.

4 Examples

Let us give here explicit formulæ for the bilinear differential operators (2.3) with $k = 1, 2$. Using (3.15) and (3.16) one has:

$$\begin{aligned} B_2 = & n\mu(2 + n(2\mu - 1))R_{\xi\xi} - (2 + n(2\mu - 1))(2 + n(2\lambda - 1))R_{\xi\eta} \\ & + n\lambda(2 + n(2\lambda - 1))R_{\eta\eta} \end{aligned}$$

and

$$\begin{aligned}
B_4 = & -(2 + n(2\lambda - 1))(2 + n(2\mu - 1))(4 + n(2\lambda - 1))(4 + n(2\mu - 1))R_{\xi\eta}^2 \\
& + 2(1 + n\mu)(2 + n(2\mu - 1))(4 + n(2\lambda - 1))(4 + n(2\mu - 1))R_{\xi\xi}R_{\xi\eta} \\
& + 2(1 + n\lambda)(2 + n(2\lambda - 1))(4 + n(2\lambda - 1))(4 + n(2\mu - 1))R_{\xi\eta}R_{\eta\eta} \\
& - \frac{1}{2} \left((2 + n(2\lambda - 1)) + 2(1 + n\mu)(2 + n\lambda) + (2 + n(2\mu - 1)) \right. \\
& \left. + 2(1 + n\lambda)(2 + n\mu) \right) (4 + n(2\lambda - 1))(4 + n(2\mu - 1))R_{\xi\xi}R_{\eta\eta} \\
& - (1 + n\mu)(2 + n(2\mu - 1))n\mu(4 + n(2\mu - 1))R_{\xi\xi}^2 \\
& - (1 + n\lambda)(2 + n(2\lambda - 1))n\lambda(4 + n(2\lambda - 1))R_{\eta\eta}^2.
\end{aligned}$$

The expressions of further orders operators are much more complicated, and we do not have an explicit general formula, except for some particular coefficients $c_{r,s,t}$. A direct computation using equations leads to

$$c_{i,k-i,0} = (-1)^i \binom{k}{i} \frac{(k-1+n\mu)(k-2+n\mu) \cdots (k-i+n\mu)}{(2+n(2\lambda-1))(4+n(2\lambda-1)) \cdots (2i+n(2\lambda-1))} c_{0,k,0}$$

and

$$c_{0,k-i,i} = (-1)^i \binom{k}{i} \frac{(k-1+n\lambda)(k-2+n\lambda) \cdots (k-i+n\lambda)}{(2+n(2\mu-1))(4+n(2\mu-1)) \cdots (2i+n(2\mu-1))} c_{0,k,0}.$$

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